

# Asymptotic Properties of the $p$ -Adic Fractional Integration Operator

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*To the blessed memory of M. L. Gorbachuk*

## Abstract

We study asymptotic properties of the  $p$ -adic version of a fractional integration operator introduced in the paper by A. N. Kochubei, Radial solutions of non-Archimedean pseudo-differential equations, *Pacif. J. Math.* **269** (2014), 355–369.

**Key words:**  $p$ -adic numbers; Vladimirov's  $p$ -adic fractional differentiation operator;  $p$ -adic fractional integration operator; asymptotic expansion

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# 1 Introduction

**1.1.** In analysis of complex-valued functions on the field  $\mathbb{Q}_p$  of  $p$ -adic numbers (or, more generally, on a non-Archimedean local field), the basic operator is Vladimirov's fractional differentiation operator  $D^\alpha$ ,  $\alpha > 0$ , defined via the Fourier transform or, for wider classes of functions, as a hypersingular integral operator [1, 5]. Properties of this  $p$ -adic pseudo-differential operator were studied by Vladimirov (see [5]) and found to be more complicated than those of its classical counterparts. For example, as an operator on  $L^2(\mathbb{Q}_p)$ , it has a point spectrum of infinite multiplicity. However, it was shown in [2] to behave much simpler on radial functions  $x \rightarrow f(|x|_p)$ .

In particular, in [2] the first author introduced a right inverse  $I^\alpha$  to the operator  $D^\alpha$  on radial functions, which can be seen as a  $p$ -adic analog of the Riemann-Liouville fractional integral of real analysis (including the case  $\alpha = 1$  of the usual antiderivative). Just as the Riemann-Liouville fractional integral is a source of many problems of analysis, that must be true for the operator  $I^\alpha$ .

In this paper we study asymptotic properties of the function  $I^\alpha f$  for a given asymptotic expansion of  $f$ ; for the asymptotic properties of Riemann-Liouville fractional integral see [3, 4, 7].

**1.2.** Let us recall the main definitions and notation used below.

Let  $p$  be a prime number. The field of  $p$ -adic numbers is the completion  $\mathbb{Q}_p$  of the field  $\mathbb{Q}$  of rational numbers, with respect to the absolute value  $|x|_p$  defined by setting  $|0|_p = 0$ ,

$$|x|_p = p^{-\nu} \text{ if } x = p^\nu \frac{m}{n},$$

where  $\nu, m, n \in \mathbb{Z}$ , and  $m, n$  are prime to  $p$ . It is well known that  $\mathbb{Q}_p$  is a locally compact topological field with the topology determined by the metric  $|x - y|_p$ , and that there are no absolute values on  $\mathbb{Q}$ , which are not equivalent to the "Euclidean" one, or one of  $|\cdot|_p$ . We will denote by  $dx$  the Haar measure on the additive group of  $\mathbb{Q}_p$  normalized by the condition  $\int_{|x|_p \leq 1} dx = 1$ .

The absolute value  $|x|_p$ ,  $x \in \mathbb{Q}_p$ , has the following properties:

$$\begin{aligned} |x|_p &= 0 \text{ if and only if } x = 0; \\ |xy|_p &= |x|_p \cdot |y|_p; \\ |x + y|_p &\leq \max(|x|_p, |y|_p). \end{aligned}$$

The latter property called the ultrametric inequality (or the non-Archimedean property) implies the total disconnectedness of  $\mathbb{Q}_p$  and unusual geometric properties. Note also the following consequence of the ultrametric inequality:

$$|x + y|_p = \max(|x|_p, |y|_p) \quad \text{if } |x|_p \neq |y|_p.$$

We will often use the integration formulas (see [1, 5, 6]):

$$\int_{|x|_p \leq p^n} |x|_p^{\alpha-1} dx = \frac{1 - p^{-1}}{1 - p^{-\alpha}} p^{\alpha n}; \text{ here and below } n \in \mathbb{Z}, \alpha > 0;$$

in particular,

$$\begin{aligned}\int_{|x|_p \leq p^n} dx &= p^n; \\ \int_{|x|_p = p^n} dx &= (1 - \frac{1}{p})p^n; \\ \int_{|x|_p = 1} |1 - x|_p^{\alpha-1} &= \frac{p - 2 + p^{-\alpha}}{p(1 - p^{-\alpha})}.\end{aligned}$$

See [1, 5] for further details of analysis of complex-valued functions on  $\mathbb{Q}_p$ .

From now on, we consider the case  $\alpha > 1$ . The integral operator  $I^\alpha$  introduced in [2] has the form

$$(I^\alpha f(x)) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \int_{|y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) f(y) dy, \quad (1)$$

where  $f$  is a locally integrable function on  $\mathbb{Q}_p$ . See [2] for its connection to the Vladimirov operator  $D^\alpha$  and applications to non-Archimedean counterparts of ordinary differential equations. Note that our results can be generalized easily to the case of general non-Archimedean local fields.

## 2 Asymptotics at the origin

Let  $0 < M_0 < M_1 < M_2 < \dots$ ,  $M_n \rightarrow \infty$ . Then the sequence  $f_n(x) = |x|_p^{M_n}$  is an asymptotic scale for  $x \rightarrow 0$  (see, for example, §16 of [4] for the main notions regarding asymptotic expansions).

**Theorem 1.** *Suppose that a function  $f$  admits an asymptotic series expansion*

$$f \sim \sum_{n=0}^{\infty} a_n |x|_p^{M_n}, \quad |x|_p \rightarrow 0, a_n \in \mathbb{C}.$$

Then

$$(I^\alpha f(x)) \sim \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \sum_{n=0}^{\infty} a_n b_n |x|_p^{M_n + \alpha}, \quad |x|_p \rightarrow 0, \quad (2)$$

where

$$b_n = \frac{p^{-\alpha+1} - 1}{(1 - p^{-\alpha})p} + (1 - p^{-1}) \sum_{k=1}^{\infty} (1 - p^{-k(\alpha-1)}) p^{-k(M_n+1)}.$$

*Proof.* We have

$$f = \sum_{n=0}^N a_n |x|_p^{M_n} + R_N(x), \quad R_N(x) = o(|x|_p^{M_N}) \quad |x|_p \rightarrow 0.$$

Then  $I^\alpha f = I_{(1)}^\alpha + I_{(2)}^\alpha$ ,

$$I_{(1)}^\alpha = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \int_{|y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) \left( \sum_{n=0}^N a_n |y|_p^{M_n} \right) dy,$$

$$I_{(2)}^\alpha = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \int_{|y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) R_N(y) dy.$$

After the change of variables  $y = sx$  we get

$$I_{(1)}^\alpha = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} |x|_p^\alpha \int_{|s|_p \leq 1} (|1 - s|_p^{\alpha-1} - |s|_p^{\alpha-1}) \left( \sum_{n=0}^N a_n |x|_p^{M_n} |s|_p^{M_n} \right) ds = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} |x|_p^\alpha (A + B)$$

where

$$\begin{aligned} A &= \int_{|s|_p < 1} (1 - |s|_p^{\alpha-1}) \left( \sum_{n=0}^N a_n |x|_p^{M_n} |s|_p^{M_n} \right) ds \\ &= \sum_{n=0}^N a_n |x|_p^{M_n} \sum_{k=1}^{\infty} (1 - p^{-k(\alpha-1)}) p^{-kM_n} \int_{|s|_p = p^{-k}} ds \\ &= (1 - p^{-1}) \sum_{n=0}^N a_n |x|_p^{M_n} \sum_{k=1}^{\infty} (1 - p^{-k(\alpha-1)}) p^{-k(M_n+1)}, \\ B &= \sum_{n=0}^N a_n |x|_p^{M_n} \int_{|s|_p = 1} (|1 - s|_p^{\alpha-1} - 1) ds = \frac{p^{-\alpha+1} - 1}{(1 - p^{-\alpha})p} \sum_{n=0}^N a_n |x|_p^{M_n}. \end{aligned}$$

On the other hand, since  $|R_N(x)| \leq C|x|_p^{M_{N+1}}$ , we find that for some constant  $C_1 > 0$ ,

$$|I_{(2)}^\alpha| \leq C_1 |x|_p^{\alpha+M_{N+1}} \int_{|s|_p \leq 1} (|1 - s|_p^{\alpha-1} - |s|_p^{\alpha-1}) |s|_p^{M_{N+1}} ds = O(|x|_p^{\alpha+M_{N+1}}).$$

The above calculations result in the asymptotic relation (2). ■

### 3 Asymptotics at infinity

For positive functions  $\varphi, \psi$ , we write  $\varphi(x) \asymp \psi(x)$ ,  $|x|_p \rightarrow \infty$ , if  $c\psi(x) \leq \varphi(x) \leq d\psi(x)$ , for large values of  $|x|_p$ ,  $x \in \mathbb{Q}_p$ , for some positive constants  $c, d$ .

**Theorem 2.** Suppose that  $a \leq f(x) \leq b$  ( $a, b > 0$ ) for  $|x|_p < 1$ ,  $|f(x)| \leq C|x|_p^{-M}$ ,  $M > 1$ ,  $C > 0$ , for  $|x|_p \geq 1$ . Then

$$(I^\alpha f)(x) \asymp |x|_p^{\alpha-1}, \quad |x|_p \rightarrow \infty. \quad (3)$$

*Proof.* Let us rewrite (1) with  $|x|_p \geq 1$  in the form  $I^\alpha f = J_{(1)}^\alpha f + J_{(2)}^\alpha f$  where

$$(J_{(1)}^\alpha f)(x) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \int_{|y|_p < 1} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) f(y) dy,$$

$$(J_{(2)}^\alpha f)(x) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} \int_{1 \leq |y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) f(y) dy$$

Then

$$(J_{(1)}^\alpha f)(x) \asymp \int_{|y|_p < 1} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) dy \asymp |x|_p^{\alpha-1}.$$

Next, if  $|x|_p = p^N$ ,  $N \geq 0$ , then

$$\begin{aligned} |(J_{(2)}^\alpha f)(x)| &\leq C \int_{1 \leq |y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) |y|_p^{-M} dy \\ &= C \left\{ \sum_{j=0}^{N-1} \int_{|y|_p = p^j} (|x|_p^{\alpha-1} - |y|_p^{\alpha-1}) |y|_p^{-M} dy + \int_{|y|_p = p^N} (|x - y|_p^{\alpha-1} - p^{N(\alpha-1)}) p^{-MN} dy \right\} \\ &= C \left\{ \left(1 - \frac{1}{p}\right) \sum_{j=0}^{N-1} p^j (p^{N(\alpha-1)} - p^{j(\alpha-1)}) p^{-Mj} \right. \\ &\quad \left. + p^{-MN} \int_{|y|_p = p^N} |x - y|_p^{\alpha-1} dy - \left(1 - \frac{1}{p}\right) p^{\alpha N - MN} \right\}. \end{aligned}$$

Calculating the integral as above and finding the sums of geometric progressions we see that  $\left| (J_{(2)}^\alpha f)(x) \right| \leq \text{const} \cdot |x|_p^{\alpha-1}$ , which proves (3).  $\blacksquare$

## 4 Logarithmic asymptotics

If a function  $f$  decays slower than it did under the assumptions of Theorem 2, then a richer asymptotic behavior is possible. Let us consider the case where  $f(t) \geq 0$ ,

$$f(x) \sim |x|_p^{-\beta} \sum_{n=0}^{\infty} a_n (\log |x|_p)^{\gamma-n}, \quad |x|_p \rightarrow \infty, \quad (4)$$

where  $0 \leq \beta < 1$ ,  $\gamma \geq 0$ ,  $a_n \in \mathbb{R}$ .

First we need some auxiliary results.

**Lemma 1.** *Let  $0 \leq f(x) = o(|x|_p^{-\lambda})$ ,  $|x|_p \rightarrow \infty$ , where  $0 < \lambda < 1$ . Then*

$$G_1(r) \stackrel{\text{def}}{=} \int_{|y|_p \leq r} f(y) dy = o(r^{1-\lambda}), \quad r \rightarrow \infty. \quad (5)$$

*Proof.* Let  $n_0 = \lfloor \log_p r \rfloor$ . Then  $p^{n_0} \leq r \leq p^{n_0+1}$ . It is known (see Section 1) that

$$\int_{|y|_p \leq p^\nu} |y|_p^{-\lambda} dy = \frac{1 - p^{-1}}{1 - p^{\lambda-1}} p^{(1-\lambda)\nu}, \quad \nu \in \mathbb{Z}, \quad (6)$$

so that

$$G_2(r) \stackrel{\text{def}}{=} \int_{|y|_p \leq r} |y|_p^{-\lambda} dy = O(r^{1-\lambda}), \quad r \rightarrow \infty. \quad (7)$$

By our assumption, for any  $n \in \mathbb{N}$ , there exists such  $r_0 = r_0(n)$  that  $f(x) < \frac{1}{n} |x|_p^{-\lambda}$  for  $|x|_p > r_0$ . Then we can write

$$\frac{G_1(r)}{G_2(r)} = \frac{G_1(r_0(n)) + (G_1(r) - G_1(r_0(n)))}{G_2(r_0(n)) + (G_2(r) - G_2(r_0(n)))} \leq \frac{G_1(r_0(n)) + \frac{1}{n} G_3(n, r)}{G_2(r_0(n)) + G_3(n, r)}$$

where

$$G_3(n, r) = \int_{r_0 \leq |y|_p \leq r} |y|_p^{-\lambda} dy.$$

It follows from (6) that  $G_3(n, r) \rightarrow \infty$ , so that

$$0 \leq \limsup_{r \rightarrow \infty} \frac{G_1(r)}{G_2(r)} \leq \frac{1}{n}$$

where  $n$  is arbitrary. Therefore

$$\lim_{r \rightarrow \infty} \frac{G_1(r)}{G_2(r)} = 0,$$

which gives, together with (7), the required asymptotic relation (5).  $\blacksquare$

**Lemma 2.** Let  $0 \leq \beta < 1$ ,  $k \in \mathbb{N}$ . For any  $\varepsilon > 0$ , such that  $\beta + \varepsilon < 1$ ,

$$K_r \stackrel{\text{def}}{=} \int_{|t|_p \leq r^{-1}} (|1 - t|_p^{\alpha-1} - |t|_p^{\alpha-1}) |t|_p^{-\beta} \log |t|_p^k dt = O(r^{\beta+\varepsilon-1}), \quad r \rightarrow \infty. \quad (8)$$

*Proof.* Assuming that  $r > 2$ , we have  $|t|_p < \frac{1}{2}$ , so that  $|1 - t|_p^{\alpha-1} - |t|_p^{\alpha-1} = 1 - |t|_p^{\alpha-1} \leq 1$ , and we find that

$$K_r \leq \int_{|t|_p \leq r^{-1}} |t|_p^{-\beta} \log |t|_p^k dt \leq \int_{|t|_p \leq r^{-1}} |t|_p^{-\beta-\varepsilon} dt,$$

if  $r$  is large enough, and the relation (8) follows from the integration formula (6).  $\blacksquare$

Now we are ready to consider the asymptotics of  $I^\alpha f$  for a function  $f$  satisfying (4). Below we use the notation

$$\binom{\gamma}{n} = \frac{\gamma(\gamma-1) \cdots (\gamma-n+1)}{n!}$$

for any real positive number  $\gamma$  and  $n \in \mathbb{N}$ .

**Theorem 3.** *If a function  $f \geq 0$  satisfies the asymptotic relation (4), then*

$$(I^\alpha f)(x) \sim \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} |x|_p^{\alpha-\beta} \sum_{n=0}^{\infty} B_n (\log |x|_p)^{\gamma-n}, \quad |x|_p \rightarrow \infty, \quad (9)$$

where

$$B_n = \sum_{k=0}^n a_{n-k} \binom{\gamma + k - n}{k} \Omega(k, \alpha, \beta),$$

$$\Omega(k, \alpha, \beta) = \int_{|t|_p \leq 1} (|1 - t|_p^{\alpha-1} - |t|_p^{\alpha-1}) |t|_p^{-\beta} (\log |t|_p)^k dt.$$

*Proof.* Let us write  $(I^\alpha f)(x)$  for  $|x|_p \geq 1$  as the sum of two integrals  $I_1$  and  $I_2$ , with the integration over  $\{y : |y|_p < |x|_p^{1/2}\}$  and  $\{y : |x|_p^{1/2} \leq |y|_p \leq |x|_p\}$  respectively.

Denote  $\mathcal{K}(x, y) = |x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}$ . Considering  $I_1$ , for  $|y|_p \leq |x|_p$ , we have

$$|\mathcal{K}(x, y)| \leq |x|_p^{\alpha-1}. \quad (10)$$

Indeed, if  $|x|_p > 1$ , then  $|y|_p < |x|_p$ ,  $\mathcal{K}(x, y) = |x|_p^{\alpha-1} - |y|_p^{\alpha-1}$ , and we get (10). If  $|x|_p = 1$ ,  $|y|_p < 1$ , then  $0 < \mathcal{K}(x, y) = 1 - |y|_p^{\alpha-1} < |x|_p^{\alpha-1}$ .

It follows from (10) that

$$0 \leq I_1 \leq C |x|_p^{\alpha-1} \int_{|y|_p < |x|_p^{1/2}} f(y) dy,$$

and by (4) and Lemma 1, for any small  $\varepsilon > 0$ ,

$$I_1 = o\left(|x|_p^{\alpha-\beta+\frac{\beta+\varepsilon-1}{2}}\right), \quad |x|_p \rightarrow \infty. \quad (11)$$

Considering  $I_2$  we write

$$f(t) = |t|_p^{-\beta} \sum_{n=0}^N a_n (\log |t|_p)^{\gamma-n} + R_N(t), \quad R_N(t) = O(|t|_p^{-\beta} (\log |t|_p)^{\gamma-N-1}) \quad |t|_p \rightarrow \infty.$$

Denote

$$L(\alpha, \beta, \gamma, x) = \int_{|x|_p^{1/2} \leq |y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1}) |y|_p^{-\beta} (\log |y|_p)^\gamma dy$$

$$= |x|_p^{\alpha-\beta} (\log |x|_p)^\gamma \int_{|x|_p^{-1/2} \leq |t|_p \leq 1} (|1 - t|_p^{\alpha-1} - |t|_p^{\alpha-1}) |t|_p^{-\beta} \left(1 + \frac{\log |t|_p}{\log |x|_p}\right)^\gamma dt$$

where on the domain of integration,

$$\left| \frac{\log |t|_p}{\log |x|_p} \right| \leq \frac{1}{2},$$

and we may write, for a non-integer  $\gamma$ , the convergent binomial series

$$\left(1 + \frac{\log |t|_p}{\log |x|_p}\right)^\gamma = \sum_{k=0}^{\infty} \binom{\gamma}{k} \left(\frac{\log |t|_p}{\log |x|_p}\right)^k.$$

Note that we can use the Taylor formula with the integral form of the remainder

$$(1+s)^\gamma = \sum_{k=0}^N \binom{\gamma}{k} s^k + \frac{\gamma(\gamma-1)\cdots(\gamma-N)}{N!} \int_0^s (1+\sigma)^{\gamma-N-1} (s-\sigma)^N d\sigma$$

where

$$\int_0^s (1+\sigma)^{\gamma-N-1} (s-\sigma)^N d\sigma = s^{N+1} \int_0^1 (1+s\tau)^{\gamma-N-1} (1-\tau)^N d\tau = s^{N+1} \int_0^1 (1+s(1-\tau))^{\gamma-N-1} \tau^N d\tau.$$

If  $-\frac{1}{2} < s < \frac{1}{2}$ ,  $0 < \tau < 1$ , then  $\frac{1}{2} \leq 1+s(1-\tau) \leq \frac{3}{2}$ . Therefore

$$\left(1 + \frac{\log |t|_p}{\log |x|_p}\right)^\gamma = \sum_{k=0}^N \binom{\gamma}{k} \left(\frac{\log |t|_p}{\log |x|_p}\right)^k + S_N(t, x),$$

$$S_N(t, x) = O\left(\left(\frac{\log |t|_p}{\log |x|_p}\right)^{N+1}\right), \quad |x|_p \rightarrow \infty,$$

and this asymptotics is uniform with respect to  $t$ ,  $|t|_p \in [|x|_p^{-1/2}, 1]$ .

Substituting and using Lemma 2 we obtain the expansion

$$L(\alpha, \beta, \gamma, x) = |x|_p^{\alpha-\beta} \sum_{k=0}^N \binom{\gamma}{k} \Omega(k, \alpha, \beta) (\log |x|_p)^{\gamma-k} + o(|x|_p^{\alpha-\beta} (\log |x|_p)^{\gamma-N}), \quad |x|_p \rightarrow \infty. \quad (12)$$

We have

$$I_2 = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \sum_{n=0}^N a_n L(\alpha, \beta, \gamma-n, x) + \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int_{|x|_p^{1/2} \leq |y|_p \leq |x|_p} (|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}) R_N(y) dy$$

where

$$\begin{aligned} & \int_{|x|_p^{1/2} \leq |y|_p \leq |x|_p} (|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1}) R_N(y) dy \\ & \leq CL(\alpha, \beta, \gamma-N-1, x) = O\left(|x|_p^{\alpha-\beta} (\log |x|_p)^{\gamma-N-1}\right), \quad |x|_p \rightarrow \infty. \end{aligned}$$

The last estimate is a consequence of (12).

Now the asymptotic relations (11) and (12) imply the required relation (9). ■

In our final result, we give a modification of Theorem 3 for the case where  $\beta = 1$ .



**Theorem 4.** Suppose that  $f$  is nonnegative,

$$f(x) \sim |x|_p^{-1} \sum_{n=0}^{\infty} a_n (\log |x|_p)^{\gamma-n}, \quad |x|_p \rightarrow \infty.$$

Then

$$(I^\alpha f)(x) \sim \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \left[ |x|_p^{\alpha-1} \int_{|y|_p \leq |x|_p} f(y) dy + \sum_{n=0}^{\infty} \tilde{B}_n (\log |x|_p)^{\gamma-n} \right], \quad |x|_p \rightarrow \infty \quad (13)$$

where

$$\begin{aligned} \tilde{B}_n &= \sum_{k=0}^n a_{n-k} \binom{\gamma+k-n}{k} \tilde{\Omega}(k, \alpha), \\ \tilde{\Omega}(k, \alpha) &= \int_{|t|_p \leq 1} (|1-t|_p^{\alpha-1} - |t|_p^{\alpha-1} - 1) |t|_p^{-1} (\log |t|_p)^k dt. \end{aligned}$$

*Proof.* Let us write  $I^\alpha f = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} (J_1 + J_2 + J_3)$  where

$$\begin{aligned} J_1 &= \int_{|y|_p \leq |x|_p^{1/2}} (|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1} - |x|_p^{\alpha-1}) f(y) dy, \\ J_2 &= \int_{|x|_p^{1/2} \leq |y|_p \leq |x|_p} (|x-y|_p^{\alpha-1} - |y|_p^{\alpha-1} - |x|_p^{\alpha-1}) f(y) dy, \\ J_3 &= |x|_p^{\alpha-1} \int_{|y|_p \leq |x|_p} f(y) dy. \end{aligned}$$

Choosing  $\varepsilon > 0$ , such that  $1 + \varepsilon < \alpha$ , we see that  $f(x) = o(|x|_p^{-1+\varepsilon})$ ,  $|x|_p \rightarrow \infty$ . By Lemma 1,

$$\int_{|y|_p \leq |x|_p^{1/2}} f(y) dy = o(|x|_p^{\frac{\varepsilon}{2}}), \quad |x|_p \rightarrow \infty.$$

For the kernel of the above integral operator we get, considering various cases, the estimate

$$||x-y|_p^{\alpha-1} - |y|_p^{\alpha-1} - |x|_p^{\alpha-1}| \leq 2|y|_p^{\alpha-1}$$

It follows from Lemma 1 that

$$|J_1| \leq 2 \int_{|y|_p \leq |x|_p^{1/2}} |y|_p^{\alpha-1} f(y) dy = o(|x|_p^{\frac{\alpha-1+\varepsilon}{2}}), \quad |x|_p \rightarrow \infty. \quad (14)$$

By our assumption,

$$f(t) = |t|_p^{-1} \sum_{n=0}^N a_n (\log |t|_p)^{\gamma-n} + R_N(t), \quad R_N(t) = O(|t|_p^{-1} (\log |t|_p)^{\gamma-N-1}), \quad |t|_p \rightarrow \infty.$$

Let us consider the expression

$$\begin{aligned}\tilde{L}(\alpha, \gamma, x) &= \int_{|x|_p^{1/2} \leq |y|_p \leq |x|_p} (|x - y|_p^{\alpha-1} - |y|_p^{\alpha-1} - |x|_p^{\alpha-1}) |y|_p^{-1} (\log |y|_p)^\gamma dy \\ &= |x|_p^{\alpha-1} \int_{|x|_p^{-1/2} \leq |t|_p \leq 1} (|1 - t|_p^{\alpha-1} - |t|_p^{\alpha-1} - 1) |t|_p^{-1} (\log |x|_p + \log |t|_p)^\gamma dt.\end{aligned}$$

It follows from the first integration formula from Section 1 that

$$\int_{|t|_p \leq |x|_p^{-1/2}} (|1 - t|_p^{\alpha-1} - |t|_p^{\alpha-1} - 1) |t|_p^{-1} (\log |t|_p)^k dt = o(|x|_p^{\frac{1-\alpha+\varepsilon}{2}}), \quad |x|_p \rightarrow \infty.$$

This implies (just as in the proof of Theorem 3) the expansion

$$\tilde{L}(\alpha, \gamma, x) \sim |x|_p^{\alpha-1} \sum_{k=0}^{\infty} \binom{\gamma}{k} (\log |x|_p)^{\gamma-k} \tilde{\Omega}(k, \alpha), \quad |x|_p \rightarrow \infty.$$

Taking into account (14), we come to (13). ■

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